

MAPPINGS WITH MAXIMAL RANK

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1 Introduction

We study Maximal Rank Maps and Riemannian Submersions $\pi : M \rightarrow B$, where M and B are Riemannian manifolds.

As essential tools in this work we are interested in equivalence relations between non-compact Riemannian manifolds given by Rough Isometries, a concept first introduced by M. Kanai [6].

Motivated by O'Neill [10] we investigated the question: when does a maximal rank map differ only by a rough isometry of M from the simplest type of Riemannian submersions, the projection $p_B : F \times B \rightarrow B$ of a Riemannian product manifold on one of its factors. Firstly, for Riemannian submersions $\pi : M \rightarrow B$ we show that, if the base manifold B is compact and connected, then the fibers F can be roughly isometrically immersed into M , and thus, M is roughly isometric to the product $F \times B$ of any fiber and the base space [**Theorem 4.1.1**]. When B is noncompact, connected and complete, and $\text{diam}(F)$ is uniformly bounded, the Riemannian Submersion π is a rough isometry, and thus, if a fixed fiber F is compact then M is roughly isometric to the product $F \times B$ of that fiber and the base space [**Theorem 4.1.2**]. Secondly, for onto maximal rank maps that are not necessarily submersions, by adding control on the length of horizontal vector lifts we have the same consequences [**Theorem 4.2.1**, **Theorem 4.2.3**]. We provide *Counterexamples* in section 4.2 to show that the assumptions made are necessary conditions.

The paper begins with background.

2 Rough Isometries and Riemannian Submersions

In this section we define some notation and provide some definitions according to M. Kanai [6] and O'Neill [10].

We will be interested in equivalence relations given by rough isometries, a concept first introduced in [6].

Definition 2.1 *A map $\varphi : M \rightarrow N$, between two metric spaces (M, δ) and (N, d) , not necessarily continuous, is called a rough isometry, if it satisfies the following two axioms:*

(RI.1) *There are constants $A \geq 1, C \geq 0$, such that,*

$$\frac{1}{A}\delta(p_1, p_2) - C \leq d(\varphi(p_1), \varphi(p_2)) \leq A\delta(p_1, p_2) + C, \quad \forall p_1, p_2 \in M$$

(RI.2) *The set $Im\varphi := \{q = \varphi(p), \forall p \in M\}$ is full in N , i.e.*

$$\exists \varepsilon > 0 : N = B_\varepsilon(Im\varphi) = \{q \in N : d(q, Im\varphi) < \varepsilon\}$$

In this case we say that $Im\varphi$ is ε -full in N .

One can easily show that if $\varphi : M \rightarrow N$ and $\psi : N \rightarrow M$ are rough isometries, then the composition $\psi \circ \varphi : M \rightarrow M$ is also a rough isometry.

We will denote by $\varphi^- : N \rightarrow M$ a rough inverse of φ , defined as follows: for each $q \in N$, choose $p \in M$ so that $d(\varphi(p), q) < \varepsilon$, and define $\varphi^-(q) := p$. We point out here that such a p exists because of the condition **(RI.2)**. φ^- is a rough isometry such that both $\delta(\varphi^- \circ \varphi(p), p)$ and $d(\varphi \circ \varphi^-(q), q)$ are bounded in $p \in M$ and in $q \in N$, respectively.

We refer to O'Neill [10] for the properties of Riemannian submersions. We start recalling their definition.

Let M^m and B^n be Riemannian manifolds with dimensions m and n , respectively, where $m \geq n$.

Definition 2.2 *A map $\pi : M \rightarrow B$ has maximal rank n if the derivative map π_* is surjective.*

According to [10], a tangent vector on M is said to be vertical if it is tangent to a fiber, horizontal if it is orthogonal to a fiber. A vector field on M is vertical if it is always tangent to fibers, horizontal if it is always orthogonal to fibers.

Since the derivative map π_*x of π is surjective for all $x \in M$, its rank is maximal. We can define the projections of the tangent space of M onto the subspaces of vertical and horizontal vectors, which we will denote respectively by $(VT)_x$ and $(HT)_x$ for each $x \in M$. In that case, we can decompose each tangent space to M into a direct orthogonal sum $T_x M = (VT)_x \oplus (HT)_x$.

Definition 2.3 *A Riemannian submersion $\pi : M \rightarrow B$ is an onto mapping satisfying the following two axioms:*

(S.1) *π has maximal rank;*

(S.2) *π_* preserves lengths of horizontal vectors.*

3 Long Curves and Their Lifts

Here we begin with background from O'Neill [10] and continue with an investigation of curves and their lifts.

Let $\pi : M \rightarrow B$ denote an onto mapping with maximal rank n between Riemannian manifolds M^m and B^n with $m \geq n$.

From the maximality of the rank of the onto mapping π we have the unique horizontal vector property:

Lemma 3.1 *Let $b \in B$. Given any $w \in T_b B$ and $x \in M$ satisfying $\pi(x) = b$, there exists a unique horizontal vector $v \in T_x M$ which is π -related to w , i.e. satisfying $v \in (HT)_x$ and $(\pi_*)_x(v) = w$.*

If, in addition, one has control from below over the length of horizontal vectors, then one has contro; from below over the distance in M . This is the essence of the following Lemma.

Lemma 3.2 *Assume that M and B are both connected and geodesically complete. Let $x, x' \in M$, $\Gamma_{min} \subset M$ be a minimal geodesic joining x to x' , and $\gamma_{min} \subset B$ be a minimal geodesic joining $\pi(x)$ to $\pi(x')$. Suppose that for all $b \in B$ and for all $x \in F_b$ there exist constants $\alpha \geq 1$ and $\beta > 0$, both independent of b and x , such that*

$$\frac{1}{\alpha} \|w\|_B - \beta \leq \|v\|_M \quad (1)$$

for all $w \in T_b B$, where v is the unique horizontal lift of w through x that we assume satisfies $\|v\|_M \leq 1$, and $\| \cdot \|_M, \| \cdot \|_B$ denote the inner product on TM and TB , respectively.

Then, $d_M(x, x') = \ell(\Gamma_{min}) \geq \frac{1}{\alpha} \ell(\gamma_{min}) - \beta = \frac{1}{\alpha} d_B(\pi(x), \pi(x')) - \beta$

Proof. Without loss of generality, we may assume that the horizontal lift $v \in (HT)_x$ of w satisfies $\|v\|_M \leq 1$, and that both parametrizations of Γ_{min} and γ_{min} are defined in the interval $[0, 1]$.

We may write

$$\Gamma'_{min}(t) = \Gamma'_V(t) \oplus \Gamma'_H(t) \in T_x M = (VT)_x \oplus (HT)_x, \forall t \in [0, 1]$$

Notice that by **Lemma 3.1**, $\Gamma'_H(t) \in T_{\Gamma_{min}(t)} M$ is the unique horizontal vector which is π -related to $\frac{d}{dt}(\pi \circ \Gamma_{min})(t)$, for each $t \in [0, 1]$. Assume that Γ is parametrized proportionally to arclength, and that $\|\Gamma'_H(t)\|_M \leq 1$.

We have,

$$\begin{aligned}
d_M(x, x') &= \ell(\Gamma_{min}) = \int_0^1 \|\Gamma'_V(t) \oplus \Gamma'_H(t)\|_M dt \geq \int_0^1 \|\Gamma'_H(t)\|_M dt \stackrel{(1)}{\geq} \\
&\geq \frac{1}{\alpha} \int_0^1 \left\| \frac{d}{dt} (\pi \circ \Gamma_{min})(t) \right\|_M dt - \beta = \frac{1}{\alpha} \ell(\pi \circ \Gamma_{min}) - \beta \stackrel{\text{min.geod.}}{\geq} \\
&\geq \frac{1}{\alpha} \ell(\gamma_{min}) - \beta \stackrel{\text{dist.}}{=} \frac{1}{\alpha} d_B(\pi(x), \pi(x')) - \beta
\end{aligned}$$

which concludes the Lemma. \square

In what follows lifts of curves are defined.

Definition 3.3 Let $\gamma : [t_1, t_2] \rightarrow B$ be a smooth embedded curve in B and $\Gamma : [t_1, t_2] \rightarrow M$ be any curve in M satisfying $\pi \circ \Gamma = \gamma$. The curve Γ is called a **lift** of γ .

If in addition, Γ is horizontal, i.e., $\Gamma'(t) \in (HT)_{\Gamma(t)}, \forall t \in [t_1, t_2]$, where $\Gamma(t_1) = x_0 \in M$ with $\gamma(t_1) = \pi(x_0)$, the curve Γ is called a **horizontal lift** of γ through x_0 . Recall that the horizontal lift of a curve in B , through a point $x_0 \in M$ is unique.

Next, we define long curves.

Definition 3.4 Let β be any positive constant. A smooth embedded curve $\gamma : [t_1, t_2] \rightarrow B$ is said to be a β -long curve if $\inf_{t_1 \leq t \leq t_2} \|\gamma'(t)\| \geq \beta$. In that case, $\ell(\gamma) \geq \int_{t_1}^{t_2} \|\gamma'(t)\| dt \geq \beta(t_2 - t_1)$. We say that a curve γ is simply a long curve if it is a β -long curve for some constant $\beta > 0$.

Let $\gamma : [t_1, t_2] \rightarrow B$ denote a smooth embedded curve and let $\Gamma : [t_1, t_2] \rightarrow M$ denote a lift of γ .

In the next two Propositions, under control from above (below) on the derivative of the maximal rank mapping π , we have control from below (above) over the length of any lift of a curve.

For instance, in **Proposition 3.5** for a long curve γ in B any of its lift Γ in M cannot be short, and **Proposition 3.6** the length of a lift Γ of a long curve γ is bounded above by the length of γ .

We denote by $\|\cdot\|_M$ and $\|\cdot\|_B$ the Riemannian norms in TM and TB , respectively.

Proposition 3.5 *Assume there are constants $\alpha \geq 1$ and $\beta > 0$ such that,*

$$\|(\pi_*)_x v\|_B \leq \alpha \|v\|_M + \beta \quad (2)$$

for all $x \in M$, for all $v \in T_x M$ satisfying $\|v\|_M \leq 1$.

If γ is any smooth β -long curve in B , then,

$$\ell(\Gamma) \geq \frac{1}{\alpha} [\ell(\gamma) - \beta(t_2 - t_1)] > 0$$

where $\ell(\Gamma)$ and $\ell(\gamma)$ denote the lengths of the curves Γ and γ , respectively.

Proof. First, we choose a parametrization proportional to arc length of $\Gamma : t \in [t_1, t_2] \rightarrow \Gamma(t) \in M$, an arbitrary lift of $\gamma \subset B$. We may assume without loss of generality that $\|\Gamma'(t)\|_M \leq 1$.

If we use $v = \Gamma'(t)$ in (2) and $\pi \circ \Gamma = \gamma$, we obtain

$$\|\gamma'(t)\|_B = \|(\pi_*)_{\Gamma(t)} \Gamma'(t)\|_B \leq \alpha \|\Gamma'(t)\|_M + \beta, \quad \forall t \in [t_1, t_2] \quad (3)$$

Finally, if we integrate (3), we get

$$\begin{aligned} \ell(\gamma) &= \int_{t_1}^{t_2} \|\gamma'(t)\|_B dt \leq \alpha \int_{t_1}^{t_2} \|\Gamma'(t)\|_M dt + \beta \int_{t_1}^{t_2} dt = \\ &= \alpha \cdot \ell(\Gamma) + \beta(t_2 - t_1) \Rightarrow \\ &\Rightarrow \ell(\Gamma) \geq \frac{1}{\alpha} [\ell(\gamma) - \beta(t_2 - t_1)] > 0 \end{aligned}$$

which proves the proposition.

That the second hand side of the last inequality above is positive follows from the assumption that γ is a long curve. Therefore, as it can be interpreted from the inequality shown, for a long curve γ any of its lift Γ cannot be short.

□

Proposition 3.6 *Let $\gamma : [t_1, t_2] \rightarrow B$ denote a smooth embedded curve and let $\Gamma : [t_1, t_2] \rightarrow M$ denote a lift of γ . For horizontal^(†) vectors $v \in TM$ only, assume that there is a universal constant $\alpha \geq 1$ such that,*

$$\|(\pi_*)_x v\|_B \geq \frac{1}{\alpha} \|v\|_M - \beta \quad (4)$$

for all $x \in M$, for all $v \in T_x M \setminus (VT)_x = (HT)_x = [\ker(\pi_)_x]^\perp$.*

If γ is a β -long curve then,

$$\ell(\Gamma) \leq \alpha [\ell(\gamma) + \beta(t_2 - t_1)]$$

where $\ell(\Gamma)$ and $\ell(\gamma)$ denote the lengths of the curves Γ and γ , respectively.

Now, in the proof of **Proposition 3.6** we will need the following Lemma: for a long curve in B , any of its lift in M is non-vertical.

Lemma 3.7 *Let $\gamma : [t_1, t_2] \rightarrow B$ be a smooth embedded curve in B and let $\Gamma : [t_1, t_2] \rightarrow M$ be a lift of γ . If γ is a long curve, then, Γ is non-vertical, i.e. there exists an interval $[t_1, t_2]$, such that,*

$$(\Gamma'(t))_H \neq 0, \quad \forall t \in [t_1, t_2]$$

where $\Gamma'(t) = (\Gamma'(t))_V \oplus (\Gamma'(t))_H \in T_{\Gamma(t)}M = (VT)_{\Gamma(t)} \oplus (HT)_{\Gamma(t)}$.

Proof. Since γ is a smooth, embedded long curve, there exists an interval let us say $[t_1, t_2]$, for which,

$$\gamma'(t) \neq 0, \quad \forall t \in [t_1, t_2]$$

Moreover, since for all $t \in [t_1, t_2]$ the restriction of the derivative map

$$(\pi_*)_{\Gamma(t)}|_{(HT)_{\Gamma(t)}} \text{ is an isomorphism,}$$

we thus obtain

$$\begin{aligned} (\pi_*)_{\Gamma(t)} (\Gamma'(t))_H &= (\pi_*)_{\Gamma(t)} \{ (\Gamma'(t))_V \oplus (\Gamma'(t))_H \} = (\pi_*)_{\Gamma(t)} \{ \Gamma'(t) \} = \gamma'(t) \neq 0 \\ &\xRightarrow{\text{isom.}} (\Gamma'(t))_H \neq 0 \end{aligned}$$

for all $t \in [t_1, t_2]$, and thus Γ is non-vertical. □

Proof. of Proposition 3.6 We first notice that because γ is a long curve, by **Lemma 3.7**, Γ is non-vertical.

If we use the horizontal vector $v = \Gamma'(t)$ in (4), we may write

$$0 \stackrel{\text{Prop 3.6}(\dagger)}{\neq} \|(\pi_*)_{\Gamma(t)} \Gamma'(t)\|_B \geq \frac{1}{\alpha} \|\Gamma'(t)\|_M - \beta, \quad \forall t \in [t_1, t_2]$$

and using $\pi \circ \Gamma = \gamma$ in the above inequality, we obtain

$$0 \neq \|\gamma'(t)\|_B \geq \frac{1}{\alpha} \|\Gamma'(t)\|_M - \beta, \quad \forall t \in [t_1, t_2]$$

which in turn implies that

$$\|\Gamma'(t)\|_M \leq \alpha (\|\gamma'(t)\|_B + \beta), \quad \forall t \in [t_1, t_2]$$

Finally, integrating the above inequality gives us

$$\begin{aligned} \ell(\Gamma) &= \int_{t_1}^{t_2} \|\Gamma'(t)\|_M dt \leq \alpha \int_{t_1}^{t_2} \|\gamma'(t)\|_B dt + \beta \alpha \int_{t_1}^{t_2} dt = \\ &= \alpha \cdot \ell(\gamma) + \beta \alpha (t_2 - t_1) = \alpha [\ell(\gamma) + \beta(t_2 - t_1)] > 0 \end{aligned}$$

which proves the proposition.

Therefore, as it can be interpreted from the above inequality, the length of a lift Γ of a long curve γ is controlled by above by the length of γ . \square

4 Riemannian Submersions, Maximal Rank Maps and Counterexamples

In this section we will explore Riemannian submersions and maximal rank maps $\pi : M \rightarrow B$ between Riemannian manifolds M and B .

Motivated by O'Neill [10], we will investigate this question: when does a maximal rank map $\pi : M \rightarrow B$ differ only by a rough isometry of M from the simplest type of Riemannian submersions, the projection $p_B : F \times B \rightarrow B$ of a Riemannian product manifold on one of its factors.

4.1 Riemannian Submersions

We first show that, if the base manifold B is compact and connected, then the fibers F can be roughly isometrically immersed into M , and thus, M is roughly isometric to the product $F \times B$ of any fiber F and the base space B [**Theorem 4.1.1**]. Secondly, when B is noncompact, connected and complete, and $\text{diam}(F)$ is uniformly bounded, we show that the Riemannian submersion $\pi : M \rightarrow B$ is a rough isometry, and thus, if a fixed fiber F is compact then M is roughly isometric to the product $F \times B$ of that fiber F and the base space B [**Theorem 4.1.2**].

Theorem 4.1.1 *Let $\pi : M \rightarrow B$ be a Riemannian submersion. Suppose B is compact and connected, and for each $b \in B$ the fiber $\pi^{-1}(b)$ has the induced metric from (M, d) . Then for each $b \in B$, the inclusion $\iota : \pi^{-1}(b) \hookrightarrow M$ is a rough isometry.*

In particular, since B is compact, M is roughly isometric to the product $\pi^{-1}(b) \times B$.

Theorem 4.1.2 *Let $\pi : M \rightarrow B$ be a Riemannian submersion, where B is connected and complete. Suppose that, for some constant $m > 0$, all fibers satisfy the universal diameter property:*

$$(\mathbf{UDF}) \quad \text{diam}(\pi^{-1}(b)) \leq m < \infty, \forall b \in B.$$

Then, $\pi : M \rightarrow B$ is a rough isometry.

In particular, if for some b_0 the fiber $\pi^{-1}(b_0)$ is compact, then M is roughly isometric to the product $\pi^{-1}(b_0) \times B$.

Note that **Theorem 4.1.1** is a Corollary of **Theorem 4.2.1**, and **Theorem 4.1.2** is a Corollary of **Theorem 4.2.3**, proven in the next section.

4.2 Non-Submersions Surjective Maximal Rank Maps

In this section, we prove that for onto smooth mappings with maximal rank $\pi : M \rightarrow B$, that are not necessarily submersions, the same results as in **Theorem 4.1.1** and **Theorem 4.1.2** hold, as long as we make extra assumptions on the subspaces of horizontal vectors by adding control from above on the length of horizontal vector lifts [**Theorem 4.2.1**, **Theorem 4.2.3**]. **Counterexamples** are provided in this section to show that if any of the assumptions are removed those results cease to follow [**Counterexample 4.2.2**, **Counterexample 4.2.4**, **Counterexample 4.2.5**].

Theorem 4.2.1 *Assume that B is compact, and for each $b \in B$, the fiber $\pi^{-1}(b) = F_b$ is endowed with the induced metric from (M, d) . Suppose that for all $b \in B$, there are constants $\alpha \geq 1$ and $\beta > 0$, independent of b such that, the following inequality holds:*

$$\|v\|_M \leq \alpha \|w\|_B + \beta \tag{5}$$

for all $x \in F_b$ and $w \in T_b B$, where $v \in (HT)_x \subset T_x M$ is the horizontal lift of w through x .

Then, for each $b \in B$, the inclusion map $\iota : F_b \hookrightarrow M$ is a rough isometry.

In particular, since B is compact, M is roughly isometric to the product $\pi^{-1}(b) \times B$.

Proof. We must verify axioms **(RI.1)** and **(RI.2)** for ι , given any $b \in B$.

Clearly axiom **(RI.1)** holds since each fiber has the induced metric.

Let us denote by (M, d_M) and (B, d_B) the Riemannian metric spaces, and let $b \in B$ be fixed.

To verify axiom **(RI.2)**, we need to prove that M is an ϵ -neighborhood of $\iota(F_b) \subseteq M$, for some $\epsilon > 0$, i.e. we must find a constant $\epsilon > 0$ for which

$$d_M(y, \iota(F_b)) < \epsilon, \quad \forall y \in M$$

Without loss of generality, we may assume that B is connected, otherwise, we can repeat, on each connected component of B , the argument that will follow.

Now, since B is compact and connected it is also complete.

Thus, for any $y \in M$ there exists a minimal geodesic γ joining $\pi(y)$ to b , which we will parametrize by

$$\gamma : [0, 1] \rightarrow B, \gamma(0) = \pi(y), \gamma(1) = b$$

Since γ has a unique horizontal lift $\Gamma_y : [0, 1] \rightarrow M$, through y , and so Γ_y connects y to the fiber F_b , we can write,

$$\begin{aligned} d_M(y, \iota(F_b)) &\stackrel{dist.}{\leq} \ell(\Gamma_y) := \int_0^1 \|\Gamma'_y\|_M dt \stackrel{(5)}{\leq} \\ &\leq \alpha \int_0^1 \|(\pi_*)_{\Gamma_y(t)} \Gamma'_y(t)\|_B dt + \beta = \\ &\stackrel{\pi \circ \Gamma_y = \gamma}{=} \alpha \int_0^1 \|\gamma'(t)\|_B dt + \beta = \alpha \cdot \ell(\gamma) + \beta \end{aligned} \quad (6)$$

Now, by the compactness of B ,

$$\text{diam } B := \sup_{b_1, b_2 \in B} \{d_B(b_1, b_2)\} < \infty$$

Moreover, since γ is a minimal geodesic joining $\pi(y)$ to b ,

$$\ell(\gamma) = d_B(\pi(y), b) \leq \text{diam } B < \infty \quad (7)$$

By substituting (7) in (6), we obtain,

$$d_M(y, \iota(F_b)) \stackrel{(6)}{\leq} \alpha \cdot \ell(\gamma) + \beta \stackrel{(7)}{\leq} \alpha \cdot (\text{diam } B) + \beta \quad (8)$$

Define $\epsilon := \alpha \cdot (\text{diam } B) + \beta$, which is a positive constant independent of y , and also of $b \in B$.

For that choice of ϵ , since $y \in M$ is arbitrary, we see that (8) is exactly axiom **(RI.2)** for the inclusion map $\iota : F_b \hookrightarrow M$.

□

In what follows, we provide a Counterexample to illustrate how assumption (5) is essential in **Theorem 4.2.1**. We show that if (5) doesn't hold for some $b \in B$, then the inclusion map $\iota : F_b \hookrightarrow M$ ceases to be a rough isometry.

Counterexample 4.2.2 *We will exhibit M, B, π satisfying all the conditions in Theorem 4.2.1 with the exception of (5), i.e.,*

For any given constants $\alpha \geq 1$ and $\beta > 0$ there exist $b \in B$, $w \in T_b B$ and $x \in F_b$ satisfying:

$$\|v\|_M > \alpha \|w\|_B + \beta \quad (9)$$

where v is the horizontal lift of w in $(HT)_x \subset T_x M$.

In this case, the inclusion map $\iota : F_b \hookrightarrow M$ is not a rough isometry.

Let M and B be the following Riemannian manifolds,

$$M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 = x_3^2 + 1\}$$

and the compact unit circle,

$$B = \mathcal{S}^1 = \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1^2 + x_2^2 = 1\}$$

where the metrics on M and B are induced by the Euclidean metric on \mathbb{R}^3 .

Let $\pi : M \rightarrow B$ be defined by,

$$\pi(x_1, x_2, x_3) = \left(\frac{x_1}{\sqrt{x_1^2 + x_2^2}}, \frac{x_2}{\sqrt{x_1^2 + x_2^2}}, 0 \right)$$

Clearly $\pi : M \rightarrow B$ is an onto smooth maximal rank map.

Firstly, we remark that (9) can be verified with a series of calculations (c.f. [1]).

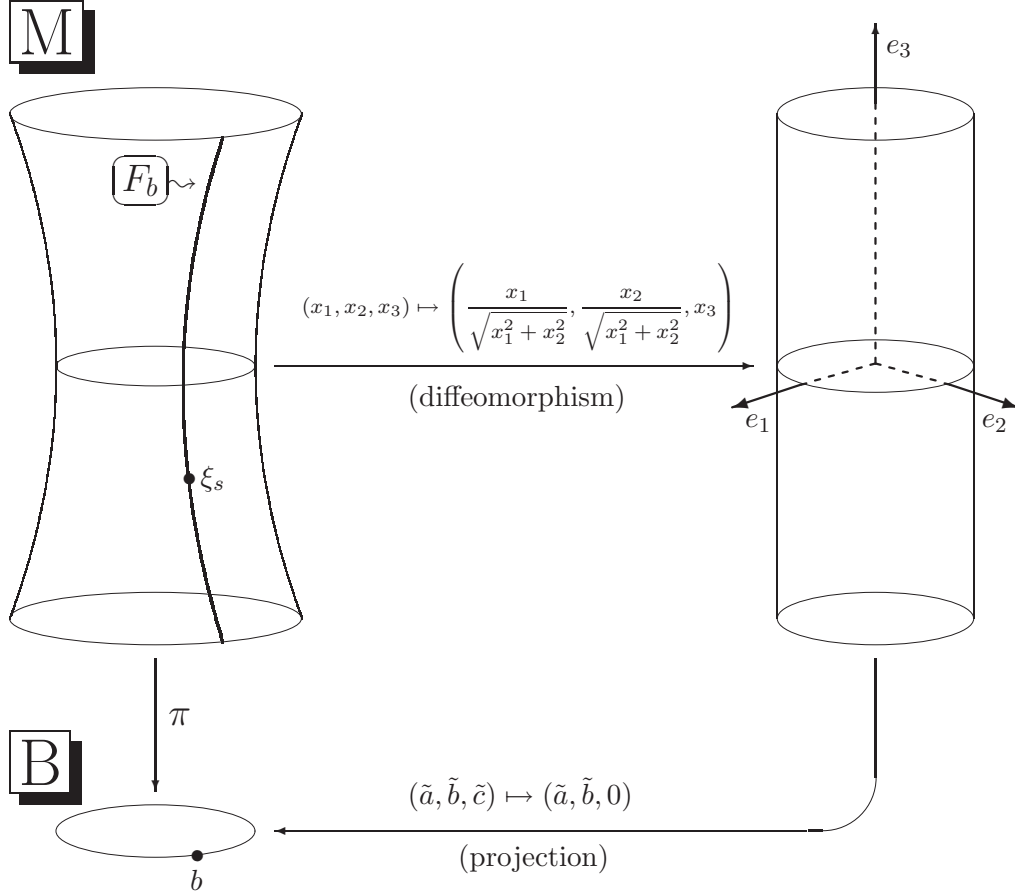
Lastly, we show that for each $b = (b_1, b_2, 0) \in B$ the inclusion $\iota : F_b \rightarrow M$ is not a rough isometry.

In that direction, we claim that **(RI.2)** fails, i.e.

$$\forall \epsilon > 0, \exists y_\epsilon = y_\epsilon(\epsilon, b) \in M, \text{ satisfying } d_M(y_\epsilon, \iota(F_b)) \geq \epsilon$$

Let γ be a compact connected smooth curve in $B = \mathcal{S}^1$, parametrized by,

$$\gamma(t) = (\cos(t), \sin(t), 0) \in B, \quad \forall t \in [0, 1]$$


 Figure 1: The map $\pi : M \rightarrow B$ in **Counterexample 4.2.2**.

with $b = \gamma(t_b)$ for some $t_b \in [0, 1]$.

A generic element in the fiber $F_b \subseteq M$ can be described as,

$$\xi_r := \left(\cos(t_b) \cdot \sqrt{r^2 + 1}, \sin(t_b) \cdot \sqrt{r^2 + 1}, r \right) \in M$$

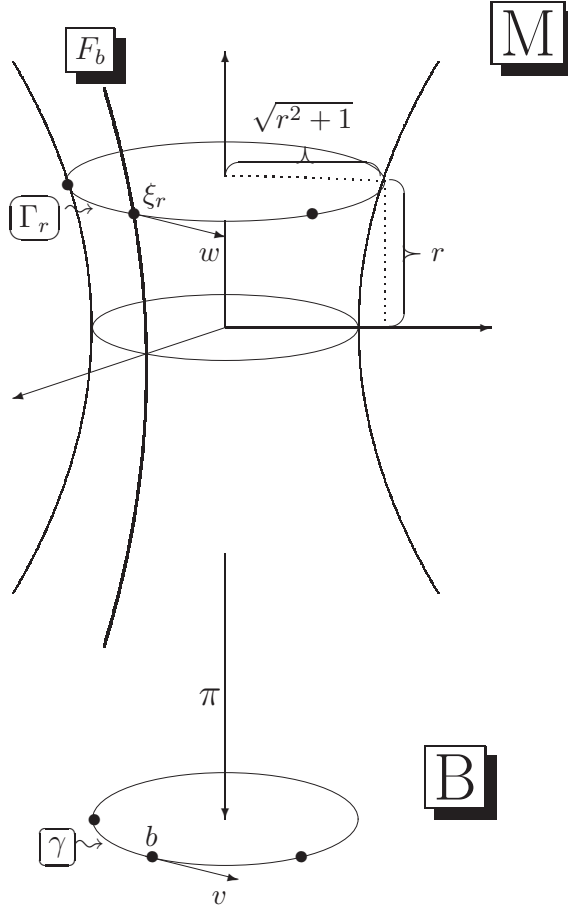
where $s \in \mathbb{R}$ is constant.

Thus, the fiber F_b , where $b = \gamma(t_b) = (\cos t_b, \sin t_b, 0)$, can be described as,

$$F_b = \{\xi_r := (\sqrt{r^2 + 1} \cdot \cos t_b, \sqrt{r^2 + 1} \cdot \sin t_b, r), r \in \mathbb{R}\}$$

It can be shown (see [1]) that the unique horizontal lift Γ_r (see Fig. 2) of γ through ξ_r , where $r > 0$, can be parametrized by,

$$\Gamma_r(t) = \left(\gamma_1(t) \cdot \sqrt{r^2 + 1}, \gamma_2(t) \cdot \sqrt{r^2 + 1}, r \right) \quad \forall t \in [0, 1]$$

Figure 2: Curve γ and its horizontal lift Γ_r .

Notice that $M \setminus F_b \neq \emptyset$.

Now, any element y_r of $M \setminus F_b$ is of the form,

$$y_r = \left(\sqrt{r^2 + 1} \cdot \cos \bar{t}, \sqrt{r^2 + 1} \cdot \sin \bar{t}, r \right)$$

for some $r \in \mathbb{R}$ and $\bar{t} \in [0, 2\pi)$, $\bar{t} \neq t_b$, where $\gamma(t_b) = (\cos t_b, \sin t_b, 0) = b \neq \pi(y_r) = (\cos \bar{t}, \sin \bar{t}, 0) = \gamma(\bar{t})$ (see Fig. 3).

We may choose $\bar{t} \in [0, 2\pi)$ as follows,

$$\bar{t} := \begin{cases} t_b + \pi, & \text{if } 0 \leq t_b < \pi \quad (:\pi \leq \bar{t} < 2\pi) \\ t_b - \pi, & \text{if } \pi \leq t_b < 2\pi \quad (:.0 \leq \bar{t} < \pi) \end{cases}$$

In particular, $\bar{t} \neq t_b$ and $|\bar{t} - t_b| = \pi$.

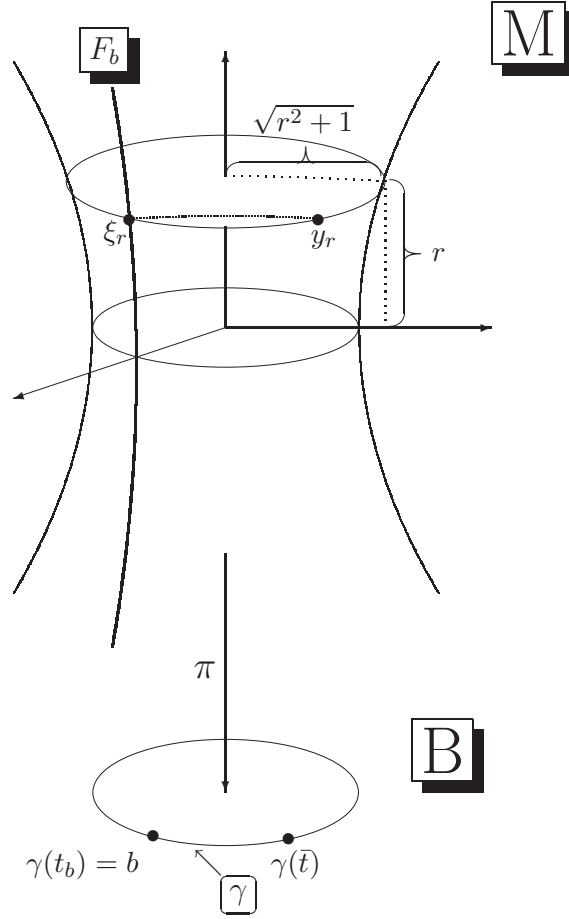


Figure 3: Generic elements ξ_r in the fiber F_b , and y_r in $M \setminus F_b$.

Moreover,

$$\begin{aligned}
 d_M(\xi_r, y_r) &= \sqrt{(r^2 + 1)(\cos t_b - \cos \bar{t})^2 + (r^2 + 1)(\sin t_b - \sin \bar{t})^2} = \\
 &= \sqrt{r^2 + 1} \sqrt{1 - 2 \cos t_b \cos \bar{t} - 2 \sin t_b \sin \bar{t} + 1} = \\
 &= \sqrt{r^2 + 1} \sqrt{2} \sqrt{1 - \cos(t_b - \bar{t})} = \sqrt{r^2 + 1} \sqrt{2} \sqrt{2} = 2\sqrt{r^2 + 1} \quad (10)
 \end{aligned}$$

Let $\epsilon > 0$ be arbitrary.

If $\epsilon \leq 2$, by (10) we have (see Fig. 4),

$$d_M(\iota(F_b), y_0) = d_M(\xi_0, y_0) \stackrel{(10)}{=} 2 \geq \epsilon$$

which shows that **[RI.2]** fails for $y_\epsilon := y_0 = (\cos \bar{t}, \sin \bar{t}, 0) \in M \setminus F_b$.

If $\epsilon > 2$, consider any $r \in \mathbb{R}$ satisfying

$$r > \frac{\sqrt{\epsilon^2 - 4}}{2} = \frac{\sqrt{\epsilon - 2}\sqrt{\epsilon + 2}}{2} > 0$$

this choice of r being possible, because property (9) holds for this Counterexample.

In that case, we have,

$$\begin{aligned} 2r > \sqrt{\epsilon^2 - 4} > 0 &\implies 4r^2 > \epsilon^2 - 4 > 0 \implies 4(r^2 + 1) > \epsilon^2 \implies \\ &\implies 2\sqrt{r^2 + 1} > \epsilon \end{aligned} \quad (11)$$

In what follows we will define (see Fig. 4),

$$y_\epsilon = \left(\sqrt{r_\epsilon^2 + 1} \cdot \cos \bar{t}, \sqrt{r_\epsilon^2 + 1} \cdot \sin \bar{t}, r_\epsilon \right) \in M \setminus F_b$$

satisfying the 2 conditions,

- $r_\epsilon > r$; and
- the unique straight line passing through y_ϵ and ξ_r is perpendicular to F_b at ξ_r , thus giving us the realization of the distance $d_M(y_\epsilon, \iota(F_b)) = d_M(y_\epsilon, \xi_r)$.

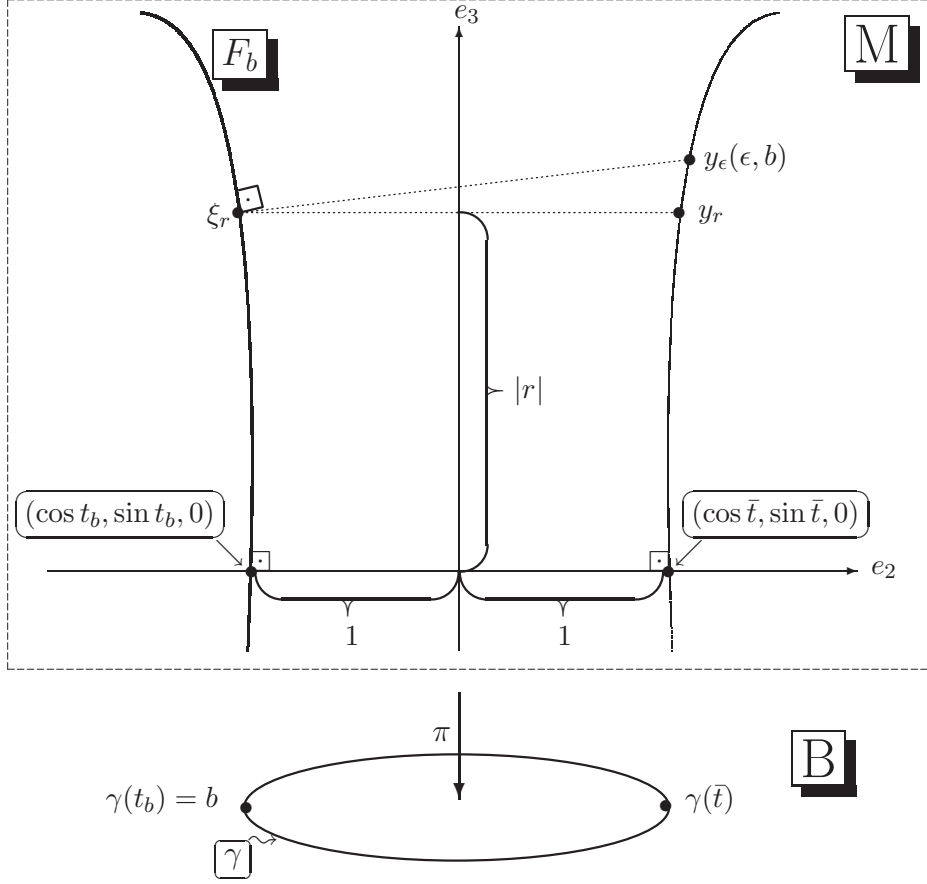
We may assume for the sake of a much simplified calculation, that, $t_b = \frac{3\pi}{2}$ and $\bar{t} = \frac{\pi}{2}$, since M is symmetric with respect to both axis e_3 and e_2

Thus we have, $\gamma\left(\frac{3\pi}{2}\right) = (0, -1, 0) = b, \gamma\left(\frac{\pi}{2}\right) = (0, 1, 0)$, a generic element $\xi_r = (0, -\sqrt{r^2 + 1}, r)$ in the fiber F_b , and a generic element $y_r = (0, \sqrt{r^2 + 1}, r)$ in M , but not in the fiber F_b .

In this case, F_b is given by $x_2 = -\sqrt{x_3^2 + 1}$, and the perpendicular line to F_b at ξ_r has equation,

$$\begin{aligned} x_2 + \sqrt{r^2 + 1} &= \frac{-1}{\mu}(x_3 - r) = \\ &= \frac{\sqrt{r^2 + 1}}{r}(x_3 - r), \quad \forall x_3 \in \mathbb{R} \end{aligned} \quad (12)$$

where $\mu = \frac{d}{du} - \sqrt{u^2 + 1} \big|_{u=r} = \frac{-2r}{2\sqrt{r^2 + 1}}$.

Figure 4: Realization of the distance between y_ϵ and the fiber F_b .

Since $y_\epsilon = (0, \sqrt{r_\epsilon^2 + 1}, r_\epsilon)$ is on the line (12), we obtain the following equation,

$$\sqrt{r_\epsilon^2 + 1} + \sqrt{r^2 + 1} = \frac{\sqrt{r^2 + 1}}{r}(r_\epsilon - r) > 0 \quad (13)$$

which defines r_ϵ .

Indeed, equation (13) has only one solution,

$$\begin{aligned} \sqrt{r_\epsilon^2 + 1} &= \frac{\sqrt{r^2 + 1}}{r}r_\epsilon - 2\sqrt{r^2 + 1} = \sqrt{r^2 + 1} \left(\frac{r_\epsilon}{r} - 2 \right) \Rightarrow \\ &\Rightarrow r_\epsilon^2 + 1 = (r^2 + 1) \left(\frac{r_\epsilon^2}{r^2} + 4 - 4\frac{r_\epsilon}{r} \right) \Rightarrow \end{aligned}$$

$$\begin{aligned}
\Rightarrow r_\epsilon^2 + 1 &= r_\epsilon^2 + \frac{r_\epsilon^2}{r^2} + 4(r^2 + 1) - 4\left(r + \frac{1}{r}\right)r_\epsilon \Rightarrow \\
&\Rightarrow \frac{1}{r^2}\mathbf{r}_\epsilon^2 - 4\left(r + \frac{1}{r}\right)\mathbf{r}_\epsilon + 4r^2 + 3 = 0 \Rightarrow \\
&\Rightarrow \mathbf{r}_\epsilon = \frac{4\left(r + \frac{1}{r}\right) \pm \sqrt{\Delta}}{\frac{2}{r^2}}
\end{aligned}$$

where

$$\begin{aligned}
\Delta &= 16\left(r + \frac{1}{r}\right)^2 - 4\frac{1}{r^2}(4r^2 + 3) = 16\left(r^2 + 2 + \frac{1}{r^2}\right) - 16 - \frac{12}{r^2} = \\
&= 16r^2 + 16 + \frac{4}{r^2} = 4\left(4r^2 + 4 + \frac{1}{r^2}\right) = 4\left(2r + \frac{1}{r}\right)^2 \Rightarrow \\
\Rightarrow \sqrt{\Delta} &= 2\left(2r + \frac{1}{r}\right) = \left(4r + \frac{2}{r}\right)
\end{aligned}$$

which implies that,

$$\mathbf{r}_\epsilon = \frac{4\left(r + \frac{1}{r}\right) \pm \left(4r + \frac{2}{r}\right)}{\frac{2}{r^2}} = 2r^3 + 2r \pm (2r^3 + r) = \begin{cases} 4^3 + 3r & \text{(solution)} \\ r & \text{(not a solution)} \end{cases}$$

Then, the only solution of equation (13) is,

$$\mathbf{r}_\epsilon = 4^3 + 3r = r(4^2 + 3) > r > 0 \tag{14}$$

Finally, by employing the expressions,

$$y_\epsilon = \left(0, \sqrt{r_\epsilon^2 + 1}, r_\epsilon\right) \quad \text{and} \quad \xi_r = (0, -\sqrt{r^2 + 1}, r)$$

and by using (13) and (14), we can estimate the distance from y_ϵ to the fiber F_b , only in terms of r ,

$$\begin{aligned}
d_M(y_\epsilon, \xi_r) &= \sqrt{0^2 + \left(\sqrt{r_\epsilon^2 + 1} + \sqrt{r^2 + 1}\right)^2 + (r_\epsilon - r)^2} \stackrel{(13)}{=} \\
&= \sqrt{\left(\frac{r^2 + 1}{r^2}\right)(r_\epsilon - r)^2 + (r_\epsilon - r)^2} = \sqrt{\left(\frac{r^2 + 1}{r^2} + 1\right)(r_\epsilon - r)^2} =
\end{aligned}$$

$$\begin{aligned}
 &= \sqrt{\frac{(2r^2+1)}{r^2}(r_\epsilon-r)^2} \stackrel{(14)}{=} \sqrt{\frac{(2r^2+1)}{r^2}(4r^3+3r-r)^2} = \\
 &= \sqrt{(2r^2+1)(4r^2+2)} = \sqrt{2r^2+1}(4r^2+2) = 2\sqrt{2r^2+1}(2r^2+1) = \\
 &= 2(2r^2+1)^{\frac{3}{2}}
 \end{aligned} \tag{15}$$

Next, we claim that,

$$(2r^2+1)^{\frac{3}{2}} > \sqrt{r^2+1} \tag{16}$$

The function $f \in C^\infty(\mathcal{R})$ defined by,

$$f(r) := (2r^2+1)^3 - (r^2+1)$$

is clearly strictly increasing on $[0, \infty)$, and thus,

$$f(r) > f(0) = 0, \quad \forall r > 0 \Rightarrow (2r^2+1)^{\frac{3}{2}} > (r^2+1)^{\frac{1}{2}}, \quad \forall r > 0$$

which is claim (16).

Now, if we combine (15), (16) and (11), we get,

$$d_M(y_\epsilon, \iota(F_b)) = d_M(y_\epsilon, \xi_r) > 2\sqrt{r^2+1} > \epsilon$$

which shows that **[RI.2]** fails for $y_\epsilon := (0, \sqrt{r_\epsilon^2+1}, r_\epsilon) \in M \setminus F_b$.

We have thus shown that for any $\epsilon > 0$ there exists $y_\epsilon \in M \setminus F_b$ for which **[RI.2]** fails. Consequently, the inclusion map $\iota : F_b \rightarrow M$ is not a rough isometry.

This describes the Counterexample.

Next, including a lower bound in assumption (5), and adding an universal diameter upper bound condition on the fibers, we will show that $\pi : M \rightarrow B$ is a rough isometry.

Theorem 4.2.3 *Let $\pi : M \rightarrow B$ be an onto smooth map with maximal rank, where B is complete. Assume the following,*

(UDF) $\exists m > 0$, a universal constant, such that $\text{diam } \{\pi^{-1}(b)\} \leq m < \infty$, for all $b \in B$; and

(HLC) $\exists \alpha \geq 1$ and $\beta > 0$ such that, for all $b \in B$ the inequality holds:

$$\frac{1}{\alpha} \|w\|_B - \beta \leq \|v\|_M \leq \alpha \|w\|_B + \beta$$

for all $x \in F_b$ and $w \in T_b B$, where $v \in (HT)_x \subset T_x M$ is the horizontal lift of w through x and we assume that v satisfies $\|v\|_M \leq 1$.

Then, $\pi : M \rightarrow B$ is a rough isometry.

In particular, if the fiber $\pi^{-1}(b_0)$ is compact for some b_0 , then M is roughly isometric to the product $\pi^{-1}(b_0) \times B$.

Proof. Firstly, note that in **(HLC)** the horizontal lift $v \in (HT)_x$ of w is assumed to satisfy $\|v\|_M \leq 1$. Otherwise, if $\|v\|_M > 1$ we define $\tilde{v} := \frac{v}{\|v\|_M}$, with the properties

- $\tilde{v} := \frac{v}{\|v\|_M} \in (HT)_x$
- $\|\tilde{v}\|_M = 1$
- $(\pi_*)_x(\tilde{v}) = \frac{w}{\|v\|_M}$

and if we use $\frac{w}{\|v\|_M}$ and \tilde{v} in **(HLC)**, we thus obtain the equivalent inequality,

$$\begin{aligned}
& \frac{1}{\alpha} \left\| \frac{w}{\|v\|_M} \right\|_B - \beta \leq \|\tilde{v}\|_M \leq \alpha \left\| \frac{w}{\|v\|_M} \right\|_B + \beta \Rightarrow \\
& \Rightarrow \frac{1}{\alpha} \frac{\|w\|_B}{\|v\|_M} - \beta \leq \frac{\|v\|_M}{\|v\|_M} \leq \alpha \frac{\|w\|_B}{\|v\|_M} + \beta \Rightarrow \\
& \Rightarrow \frac{1}{\alpha} \|w\|_B - \beta \|v\|_M \leq \|v\|_M \leq \alpha \|w\|_B + \beta \|v\|_M \Rightarrow \\
& \Rightarrow \frac{1}{\alpha} \|w\|_B \leq (\beta + 1) \|v\|_M \wedge (1 - \beta) \|v\|_M \leq \alpha \|w\|_B \Rightarrow \\
& \Rightarrow \begin{cases} \frac{1}{\alpha(\beta + 1)} \|w\|_B \leq \|v\|_M \leq \frac{\alpha}{(1 - \beta)} \|w\|_B, & \text{if } \beta \neq 1 \\ \frac{1}{\alpha(\beta + 1)} \|w\|_B \leq \|v\|_M, & \text{if } \beta = 1 \end{cases}
\end{aligned}$$

for $w \in T_b B$, where v is the unique horizontal lift of w through x with $\|v\|_M > 1$.

We must verify the validity of **(RI.1)** and **(RI.2)**.

Clearly, axiom **(RI.2)** holds since π is onto.

To verify **(RI.1)**, let $x, y \in M$.

We may assume that B is connected. Otherwise, we repeat the argument which will be utilized in this proof, on each connected component and the result will follow.

Because B is complete, there exists a minimal geodesic γ joining $\pi(x)$ to $\pi(y)$, with $\ell(\gamma) = d_B(\pi(x), \pi(y))$, which we parametrize by $\gamma : [0, 1] \rightarrow B$, where, $\gamma(0) := \pi(x), \gamma(1) := \pi(y)$.

Recall that γ has a unique horizontal lift $\Gamma_x : [0, 1] \rightarrow M$, through x , so Γ_x intersects the fiber $F_{\pi(y)}$ containing y .

We may assume, without loss of generality, that Γ_x is parametrized proportionally to arc length and $\|\Gamma_x'(t)\|_M \leq 1$ for all $t \in [0, 1]$.

Thus we can write,

$$\begin{aligned} \ell(\Gamma_x) &= \int_0^1 \|\Gamma_x'\|_M dt \stackrel{(HLC)}{\leq} \alpha \int_0^1 \|(\pi_*)_{\Gamma_x(t)} \Gamma_x'(t)\|_B dt + \beta \stackrel{\pi \circ \Gamma_x}{=} \\ &= \alpha \int_0^1 \|\gamma'(t)\|_B dt + \beta = \alpha \cdot \ell(\gamma) + \beta \stackrel{\text{def } \gamma}{=} \\ &= \alpha \cdot d_B(\pi(x), \pi(y)) + \beta \end{aligned} \quad (17)$$

By the triangle inequality, by hypothesis and the above, we have,

$$\begin{aligned} d_M(x, y) &\stackrel{\triangle}{\leq} d_M(x, \Gamma_x(1)) + d_M(\Gamma_x(1), y) \stackrel{\text{dist.}}{\leq} \\ &\leq \ell(\Gamma_x) + d_M(\Gamma_x(1), y) \stackrel{(\mathbf{UDF})}{\leq} \ell(\Gamma_x) + m \stackrel{(17)}{\leq} \\ &\leq \alpha \cdot d_B(\pi(x), \pi(y)) + \beta + m \end{aligned}$$

which can be rewritten as,

$$d_B(\pi(x), \pi(y)) \geq \frac{1}{\alpha} d_M(x, y) - \frac{(\beta + m)}{\alpha} \quad (18)$$

Now, we claim that for γ , the minimal geodesic joining $\pi(x)$ to $\pi(y)$, its length $\ell(\gamma)$ satisfies,

$$d_B(\pi(x), \pi(y)) = \ell(\gamma) \leq \alpha \cdot \ell(\varsigma) + \alpha \cdot \beta \quad (19)$$

for any smooth curve $\varsigma : [0, 1] \rightarrow M$, joining x to y .

First, observe that for any orthogonal vectors U and W ,

$$\|U \oplus W\|^2 = \|U\|^2 + \|W\|^2 \geq \max\{\|U\|^2, \|W\|^2\}$$

Now, since each tangent vector is the direct sum of a horizontal and a vertical vector, we can write,

$$\ell(\varsigma) = \int_0^1 \|\varsigma'(t)\|_M dt = \int_0^1 \|\varsigma'_H(t) \oplus \varsigma'_V(t)\|_M dt \geq \int_0^1 \|\varsigma'_H(t)\|_M dt \quad (20)$$

where we are assuming here that ς_H is parametrized proportional to arclength, and $\|\varsigma'_H(t)\|_M \leq 1, \forall t \in [0, 1]$.

Since, $(VT)_x = \ker(\pi_*)_x, \forall x \in M$, we have,

$$\ell(\pi \circ \varsigma) = \int_0^1 \|(\pi_*)_{\varsigma(t)} \varsigma'(t)\|_B dt = \int_0^1 \|(\pi_*)_{\varsigma(t)} \varsigma'_H(t)\|_B dt \quad (21)$$

From the left-hand side of **(HLC)**,

$$\begin{aligned} & \frac{1}{\alpha} \cdot \|(\pi_*)_{\varsigma(t)} \varsigma'_H(t)\|_B - \beta \leq \|\varsigma'_H(t)\|_M \Rightarrow \\ \Rightarrow & \|(\pi_*)_{\varsigma(t)} \varsigma'_H(t)\|_B \leq \alpha \cdot \|\varsigma'_H(t)\|_M + \alpha \cdot \beta \end{aligned} \quad (22)$$

for all $t \in [0, 1]$.

If we combine (20), (21) and (22), we get,

$$\begin{aligned} \ell(\pi \circ \varsigma) & \stackrel{(21)}{=} \int_0^1 \|(\pi_*)_{\varsigma(t)} \varsigma'_H(t)\|_B dt \stackrel{(22)}{\leq} \alpha \int_0^1 \|\varsigma'_H(t)\|_M + \alpha \beta \leq \\ & \stackrel{(20)}{\leq} \alpha \ell(\varsigma) + \alpha \beta \end{aligned} \quad (23)$$

Inequality (23) and the fact that γ is a minimal geodesic joining $\pi(x)$ to $\pi(y)$ imply that we can finally write,

$$d_B(\pi(x), \pi(y)) = \ell(\gamma) \leq \ell(\pi \circ \varsigma) \leq \alpha \ell(\varsigma) + \alpha \beta$$

for any smooth curve $\varsigma : [0, 1] \rightarrow M$, joining x to y , which is claim (19).

We recall that by definition of infimum, $d_M(x, y)$ is the greatest lower bound for $\{\ell(\varsigma), \text{ where } \varsigma : [0, 1] \rightarrow M \text{ is any smooth curve joining } x \text{ to } y\}$, and since ς is arbitrary in (19), we obtain,

$$d_B(\pi(x), \pi(y)) \leq \alpha \cdot d_M(x, y) + \alpha \cdot \beta \quad (24)$$

Let $A := \alpha \geq 1$ and $C := \max \left\{ \frac{\beta + m}{\alpha}, \alpha \cdot \beta \right\} > 0$.

If we now, rewrite (18) and (24) in terms of A and C , as follows,

$$\begin{aligned} \frac{1}{A} d_M(x, y) - C & \leq \frac{1}{\alpha} d_M(x, y) - \frac{(\beta + m)}{\alpha} \stackrel{(18)}{\leq} \\ & \stackrel{(24)}{\leq} d_B(\pi(x), \pi(y)) \leq \alpha d_M(x, y) + \alpha \beta \leq A d_M(x, y) + C \end{aligned}$$

we obtain **(RI.1)** for π .

□

In the following two Counterexamples, we show that the universal diameter property **(UDF)** of the fibers, and the control over the length of horizontal lifts **(HLC)** of tangent vectors are both necessary conditions in **Theorem 4.2.3**.

Counterexample 4.2.4 We will exhibit M, B, π , where B is connected and each fiber F_b is compact for all $b \in B$, satisfying all but condition **(HLC)** in **Theorem 4.2.3**, i.e.,

For any given constants $\alpha \geq 1$ and $\beta > 0$, there exist $\bar{b} \in B$, $\bar{x} \in F_{\bar{b}}$, $\bar{w} \in T_{\bar{b}}B$ such that either one of the following holds:

$$\|\bar{v}\|_M > \alpha \|\bar{w}\|_B + \beta \quad \text{or} \quad \|\bar{v}\|_M < \frac{1}{\alpha} \|\bar{w}\|_B - \beta \quad (25)$$

where \bar{v} is the unique horizontal lift of \bar{w} through \bar{x} .

In this case, the map $\pi : M \rightarrow B$ is not a rough isometry.

Let $M = \{(x, y, z) \in \mathbb{R}^3 : x^2 + z^2 = 1, y \in \mathbb{R}\}$ and $B = \mathbb{R}$, a complete and connected Riemannian manifold.

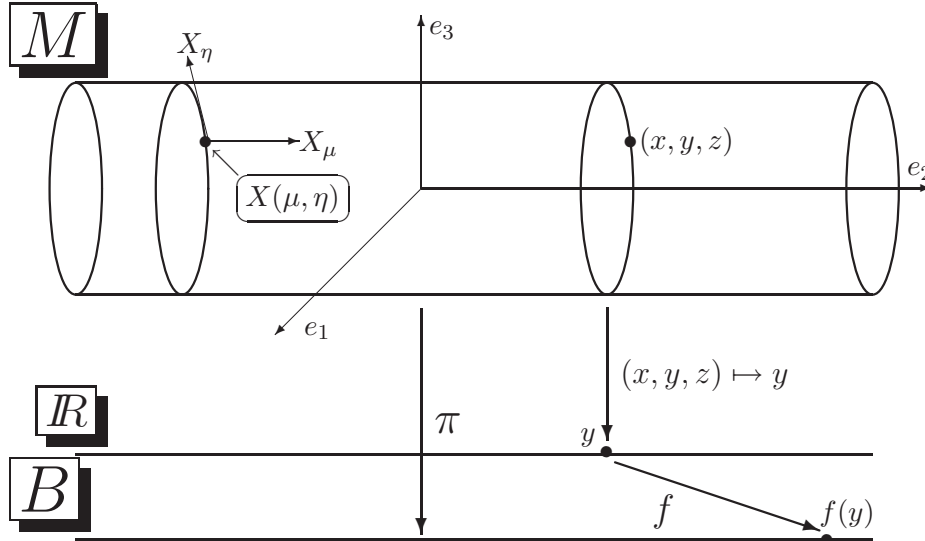
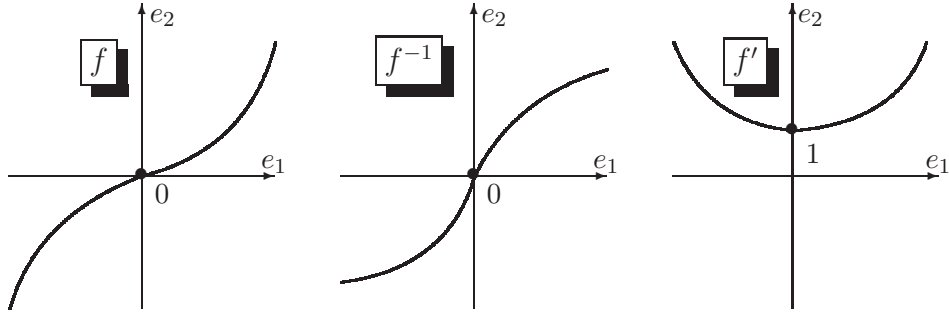


Figure 5: Manifolds M, B and the map π in **Counterexample 4.2.4**.

We first define an auxiliary C^1 -diffeomorphism $f : \mathbb{R} \rightarrow \mathbb{R}$ by,

$$f(y) = \begin{cases} e^y - 1 & \in [0, \infty) & \text{if } y \geq 0 \\ 1 - e^{-y} & \in (-\infty, 0] & \text{if } y \leq 0 \end{cases} \quad (26)$$

Figure 6: Graphs of f, f^{-1}, f' .

Let $\pi : M \rightarrow B$ be given by $\pi(x, y, z) := f(y)$.

The map π is onto and \mathcal{C}^∞ , since both the projection $(x, y, z) \mapsto y$ and f have those properties. The rank of π is maximal and no **(HLC)** is easily verified (see [1]).

Notice that the fibers have either form,

$$F_b = \pi^{-1}(b) = \begin{cases} \{(x, \ln(b+1), z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}, & \text{if } b \geq 0 \\ \{(x, -\ln(1-b), z) \in \mathbb{R}^3 : x^2 + z^2 = 1\}, & \text{if } b < 0 \end{cases}$$

Therefore, each fiber F_b is compact and $\text{diam } F_b \leq m$, for all $b \in B$, where $m = 3 > 0$ is the universal upper bound for the fibers' diameters.

Finally, we claim that π does not satisfy **(RI.1)**.

It suffices to verify that **(RI.1)** fails for π , for particular pairs of elements in M . We will show that $\forall A \geq 1, \forall C > 0, \exists y_{AC} \in \mathbb{R}$, a positive number such that,

$$\begin{aligned} d_B(\pi(x, 0, z), \pi(x, y, z)) &> A \cdot d_M((x, 0, z), (x, y, z)) + C \Leftrightarrow \\ \Leftrightarrow |f(0) - f(y)| &> A \cdot y + C \Leftrightarrow e^y - 1 > A \cdot y + C \end{aligned} \quad (27)$$

for all $y > y_{AC}$, where $x, z \in \mathbb{R} : x^2 + z^2 = 1$ are arbitrary.

Fixing constants $A \geq 1$ and $C > 0$, introduce $g \in \mathcal{C}^\infty(\mathbb{R})$, by

$$g : y \mapsto g(y) := e^y - 1 - Ay - C$$

One can show that,

$$\exists y_{AC} > 0 : \forall y > y_{AC} \Rightarrow g(y) > 0 \quad (28)$$

using the functional behavior of g (see [1]).

Therefore, (27) holds and the claim follows, and consequently π is not a rough isometry.

This describes the Counterexample.

Counterexample 4.2.5 We will exhibit M, B, π , where B is connected, satisfying all the conditions in Theorem 4.2.3, with the exception of **(UDF)**, i.e.,

The fibers' diameters are not uniformly bounded, in other words:

$$\forall m > 0, \exists b_m \in B : \text{diam } F_{b_m} > m$$

In this case, the map $\pi : M \rightarrow B$ is not a rough isometry.

Proof. Let $M = \{(0, y, z) \in \mathbb{R}^3\} \cong \{0\} \times \mathbb{R}^2$ and $B = \mathbb{R}$, a complete and connected Riemannian manifold.

Let $\pi : M \rightarrow B$ be the projection $\pi(x, y, z) := y$.

The map π is onto, \mathcal{C}^∞ , and π has maximal rank=1, and **(HLC)** is easily verified (see [1]).

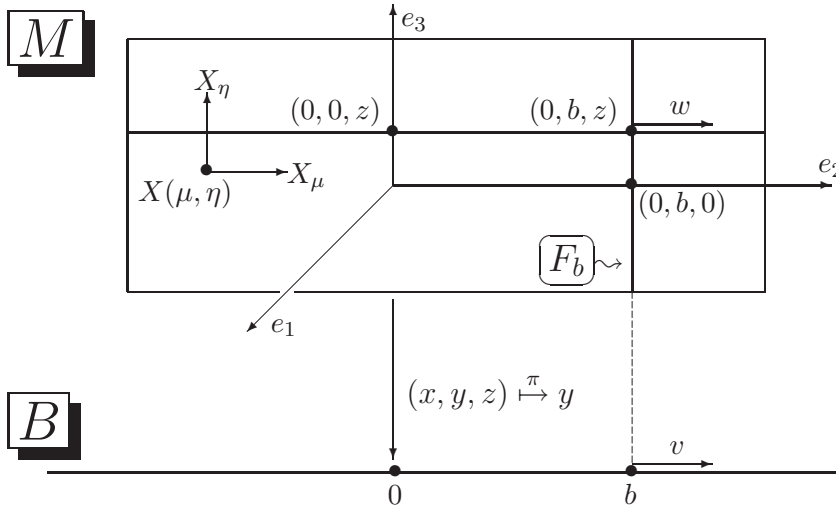


Figure 7: Manifolds M, B and the map π in **Counterexample 4.2.5**.

Each fiber is given by,

$$F_b = \pi^{-1}(b) = \{(0, b, z) : z \in \mathbb{R}\}, \quad b \in B = \mathbb{R}$$

which is a line passing through $(0, b, 0)$, determined by the intersection of M with the plane $y = b$. Hence each fiber is not compact as a subset of \mathbb{R}^3 , they all have infinite diameter, and therefore the fibers' diameters are not uniformly bounded.

Our goal next is to show that π is not a rough isometry.

It suffices to verify that π does not satisfy **(RI.1)** for particular pairs of elements in M , i.e.,

$$\forall A \geq 1, \forall C > 0, \exists \eta_{AC} \in \mathbb{R} \setminus \{0\},$$

$$\begin{aligned} d_B(\pi \circ X(\mu, \eta), \pi \circ X(\mu, 0)) &< \frac{1}{A} \cdot d_M(X(\mu, \eta), X(\mu, 0)) - C \Leftrightarrow \\ \Leftrightarrow |\mu - \mu| &< \frac{1}{A} \cdot d_M((0, \mu, \eta), (0, \mu, 0)) - C \Leftrightarrow \\ \Leftrightarrow 0 &< \frac{1}{A} \cdot |\eta| - C, \quad \forall \eta \geq \eta_{AC} \end{aligned} \quad (29)$$

Let $A \geq 1, C > 0$ be arbitrary, and define the real positive number $\eta_{AC} := AC + 1 > 0$.

We see that,

$$\eta_{AC} = AC + 1 > AC \Leftrightarrow \frac{1}{A}\eta_{AC} > C \Leftrightarrow \frac{1}{A}\eta_{AC} - C > 0 \quad (30)$$

and since, for all $\eta \geq \eta_{AC}$,

$$\frac{1}{A}\eta - C \geq \frac{1}{A}\eta_{AC} - C \stackrel{(30)}{>} 0$$

inequality (29) is verified.

Therefore, (29) holds and **(RI.1)** fails for π , which shows that π is not a rough isometry.

This describes the Counterexample.

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